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Upper and lower bounds for the connective constants of self-avoiding walks on the Archimedean and Laves lattices

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Abstract

We give improved upper and lower bounds for the connective constants of self-avoiding walks on a class of lattices, including the Archimedean and Laves lattices. The lower bounds are obtained by using Kesten's method of irreducible bridges, with an appropriate generalization for weakly regular lattices. The upper bounds are obtained as the largest eigenvalue of a certain transfer matrix. The obtained bounds show that, in the studied class of lattices, the connective constant is increasing in the average degree of the lattice. We also discuss an alternative measure of average degree.

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1. Introduction

Self-avoiding walks on lattices is a classical combinatorial problem in statistical physics; see [15] for a survey.

In this work we study the connective constants of self-avoiding walks on a class of lattices, the ALB lattices, containing the Archimedean lattices, their duals, the Laves lattices, and the Bow-tie lattice and its dual. We give upper and lower bounds for the connective constants on these lattices, improving previous bounds or providing the first bounds in most cases. Bounds for the hexagonal lattice were treated separately by Alm and Parviainen [3]. Recently, good lower bounds were obtained by Jensen [9] for several lattices. See table 1 for a summary of the best known bounds.

1.1. Self-avoiding walks

A *walk* of length n on a lattice is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, e_2, \ldots, e_n, v_n\}$ such that the edge e_i connects the vertices v_{i-1} and v_i. The walk is *self-avoiding* if all vertices $v_0, v_1, \ldots, v_n$ are distinct.

Table 1. Summary of lower and upper bounds for the ALB lattices.

Lattice	Degree	$\tilde{q}$	Lower	Estimate	Upper
$D(3.12^2)$	6	8.20	5.377 158	5.595	5.734 24 [5]
$D(4.6.12)$	6	6.82	4.463 058	4.624	4.787 227
$D(4.8^2)$	6	6.47	4.304 718	4.442	4.565 362
(3^6)	6	6	4.118 935 [9]	4.150 797 [10]	4.251 419
Bow-tie	5	5.12	3.357 574	3.445 5	3.525 448
$(3^2.4.3.4)$	5	5	3.285 284	3.374	3.451 433
$(3^3.4^2)$	5	5	3.266 402	3.350	3.425 364
$(3^4.6)$	5	5	3.206 403	3.293	3.369 117
$D(3.6.3.6)$	4	4.24	2.704 239	2.761	2.817 739
$D(3.4.6.4)$	4	4.24	2.693 424	2.763	2.828 174
(4^4)	4	4	2.625 622 [9]	2.638 159 [7]	2.679 193 [18]
(3.4.6.4)	4	4	2.511 254	2.564	2.610 835
(3.6.3.6)	4	4	2.548 497 [9]	2.560 577 [9]	2.590 305 [5]
$D(3^4.6)$	10/3	3.54	2.154 816	2.193	2.235 067
$D(3^3.4^2)$	10/3	3.41	2.112 899	2.152	2.186 720
$D(3^2.4.3.4)$	10/3	3.37	2.092 579	2.132	2.168 320
D(Bow-tie)	10/3	3.37	2.076 706	2.111	2.145 304
(6^3)	3	3	1.841 925 [9]	1.847 759 [16]	1.868 832 [3]
(4.8^2)	3	3	1.804 596 [9]	1.808 830 [12]	1.829 254
(4.6.12)	3	3	1.763 766	1.787 1	1.809 064
(3.12^2)	3	3	1.708 758 [9]	1.711 041 [12]	1.719 254 [3]

For a vertex-transitive graph, where all vertices are equivalent, let $f(n)$ denote the number of self-avoiding walks, starting at a fixed vertex.

Among general graphs, we will only consider weakly regular graphs with a finite number, K, of vertex classes. Two vertices belong to the same vertex class if they have the same number of self-avoiding walks of all lengths. For these graphs, let $f_i(n)$ denote the number of self-avoiding walks, starting at a fixed vertex in vertex class i, $i = 1, \ldots, K$.

Hammersley [6] proved that, for a class of lattices called *crystals* containing all lattices studied in this paper, there exists a constant μ, called the *connective constant*[1] of the lattice, such that

$$\lim_{n\to\infty} f_i^{1/n}(n) = \mu, \quad \text{for all} \quad i = 1, \ldots, K.$$

From the proof of this, it also follows that

$$\mu \leqslant \max_{1\leqslant i\leqslant K} f_i^{1/n}(n), \quad \text{for all} \quad n,$$

which is the basis for all upper bounds for connective constants.

[1] To be precise, Hammersley defined the connective constant as $\kappa = \log \mu$.

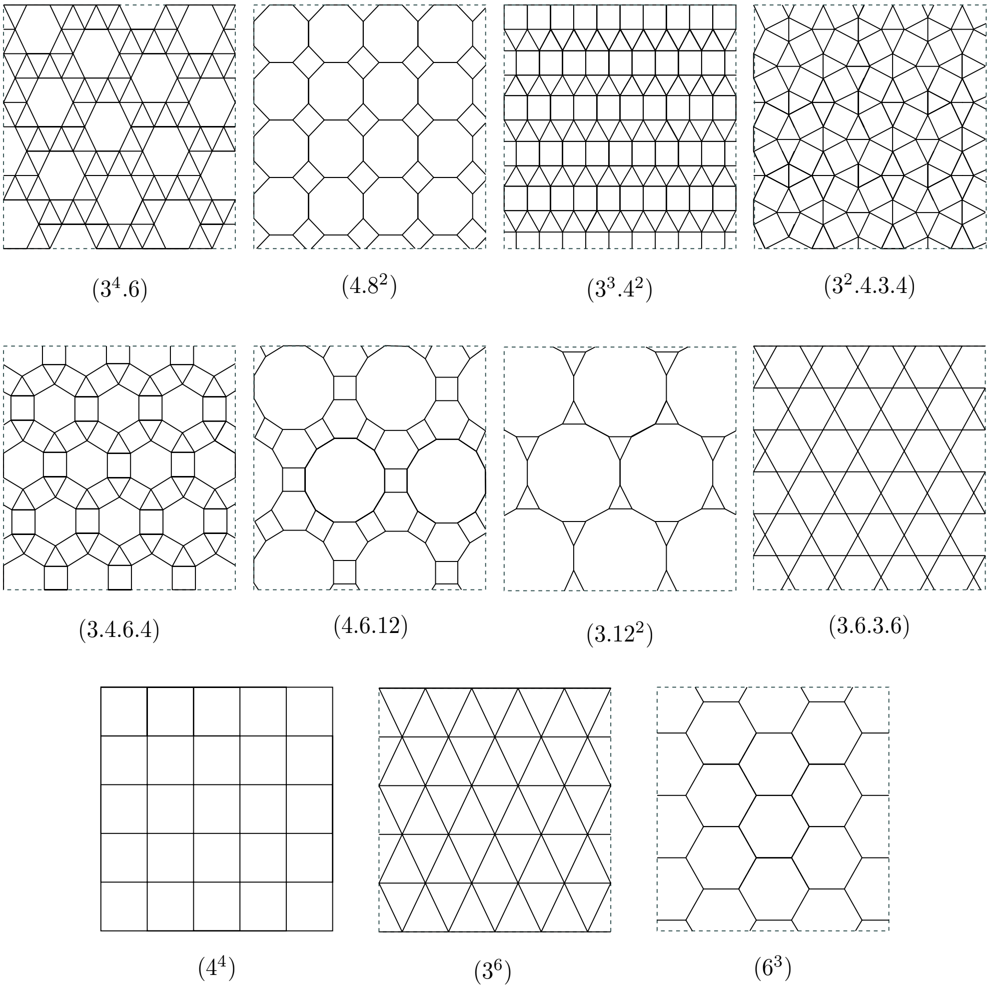

Figure 1. The Archimedean lattices.

The connective constant is unknown for all non-trivial lattices, except the hexagonal, where Nienhuis [16] has presented strong evidence that $\mu_{\text{HEX}} = \sqrt{2+\sqrt{2}} \approx 1.847\,759$. Since Jensen and Guttmann [12] have given a functional relation, (2), between the connective constant of the (3.12^2) lattice, see section 2 for a description of the lattice, and μ_{HEX}, Nienhuis' result also gives the value for $\mu_{(3.12^2)} \approx 1.711\,041$.

2. The ALB lattices

A *regular tiling* is a tiling of the plane which consists entirely of regular polygons. A vertex-transitive graph of such a regular tiling is called an *Archimedean lattice*. There are 11 such graphs, shown in figure 1. They are denoted according to a notation given in Grünbaum and Shephard [4].

When the tiling consists of only one type of regular polygon, the corresponding lattice is also *edge transitive*. Three of the Archimedean lattices are of this type, based on triangles,

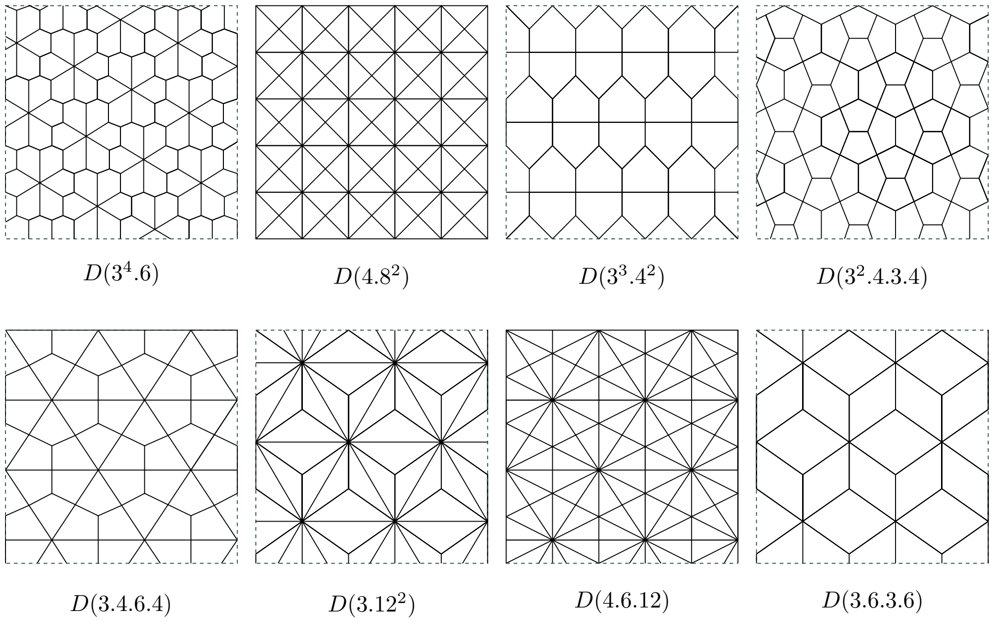

Figure 2. The Laves lattices.

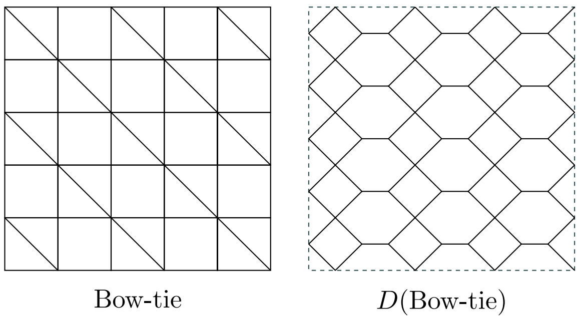

Figure 3. The Bow-tie lattice and its dual.

(3^6), squares, (4^4), or hexagons, (6^3). These lattices are often referred to as regular lattices. The remaining eight Archimedean lattices are semi-regular based on tilings with more than one type of regular polygons.

Whether a lattice is edge transitive or not will be of importance when studying both upper and lower bounds for the connective constants.

The dual of a graph G will be denoted $D(G)$. The square lattice (4^4) is self-dual; the triangular (3^6) and hexagonal (6^3) lattices are each other's duals. The duals of the eight remaining, semi-regular, Archimedean lattices constitute the class of *Laves lattices*, in which there are more than one vertex class. They are shown in figure 2.

The Laves lattices serve well as test graphs when studying how well average degree explains the connectivity of the lattice, e.g. in terms of connective constants. To get a slightly richer class, we will also include the Bow-tie lattice and its dual, see figure 3, which have similar properties to the Laves lattices.

The class of Archimedean lattices, Laves lattices, the Bow-tie lattice and its dual will be called the *ALB lattices*. All lattices in this class are *weakly regular* in the sense that they have a finite number of vertex classes under translation.

3. Lower bounds

In [13], Kesten presents a method of obtaining lower bounds for the connective constant, based on the so-called *irreducible bridges*. The method was presented for the square lattice (and its higher-dimensional analogues), but works equally well for the triangular lattice and, with a slight modification, also for the hexagonal lattice.

First, in section 3.1, we give a brief description of Kesten's original method and then, in section 3.2, we extend it to the case of weakly regular lattices.

3.1. Kesten's method for regular lattices

Given a fixed embedding of the lattice in the plane, let the coordinates for a vertex v be denoted by $(v(x), v(y))$. A *bridge* of length n is a self-avoiding walk such that

$$v_0(x) < v_i(x) \leqslant v_n(x), \quad \text{for} \quad i = 1, \ldots, n-1.$$

The idea behind this definition is that joining two bridges always produces a new bridge.

Denote the number of bridges of length n by b_n, and the generating function for bridges by ($b_0 = 1$)

$$B(t) = \sum_{n=0}^{\infty} b_n t^n.$$

An *irreducible bridge* is a bridge that cannot be decomposed into two bridges. Denote the number of irreducible bridges of length n by a_n, and the generating function for irreducible bridges by ($a_0 = 0$)

$$A(t) = \sum_{n=1}^{\infty} a_n t^n.$$

As $a_n \geqslant 0$ and $b_n \geqslant 0$ for all n, both $A(t)$ and $B(t)$ are increasing in $t > 0$.

Kesten proved that the connective constants for bridges and irreducible bridges are the same as for self-avoiding walks,

$$\lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} f^{1/n}(n) = \mu.$$

Further, $A(t)$ and $B(t)$ are related by

$$B(t) = \frac{1}{1 - A(t)},$$

so that the radius of convergence of $B(t)$ is given by

$$\frac{1}{\mu} = \sup\{t : A(t) < 1\}.$$

Thus, $A(t_0) > 1$ implies $1/\mu < t_0$, or $\mu > 1/t_0$. Further, with

$$A_N(t) = \sum_{n=1}^{N} a_n t^n,$$

we obviously have $A_N(t) \leqslant A(t)$ for all N, so that $A_N(t_0) > 1$ implies $A(t_0) > 1$ and $\mu > 1/t_0$, which provides a practical method of obtaining lower bounds for μ.

3.2. A generalization of Kesten's method to weakly regular lattices

Consider a fixed embedding of the lattice in the plane and define bridges and irreducible bridges as above. In order to be able to join two bridges into one longer bridge, we need to keep track of the vertex classes of the starting and ending vertices of the bridges.

Define a bridge of class (i, j) as a bridge that starts in a vertex of class i and ends in a vertex of class j. Then, a bridge of length m of class (i, j) can be joined with a bridge of length n of class (j, k) to form a bridge of length $n + m$ of class (i, k).

Remark 1. The introduction of a coordinate system may have the effect that we have to introduce more vertex classes than above. Two nodes are equivalent if they can be mapped on each other by a translation or by vertical reflection, preserving the lattice. See section 3.3 for more details.

Let $b_{ij}(n)$ be the number of n-step bridges of class (i, j) and $a_{ij}(n)$ be the number of n-step irreducible bridges of class (i, j), for $n \geqslant 1$. Further, let $a_{ij}(0) = 0$ for all i and j and

$$b_{ij}(0) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Then, as every bridge can be partitioned into an irreducible bridge and a bridge (possibly empty),

$$b_{ij}(n) = a_{ij}(n) + \sum_{k=1}^{n-1} \sum_{r=1}^{K} a_{ir}(k) \cdot b_{rj}(n-k) = \sum_{k=1}^{n} \sum_{r=1}^{K} a_{ir}(k) \cdot b_{rj}(n-k). \tag{1}$$

Further, introduce the generating functions

$$B_{ij}(t) = \sum_{n=0}^{\infty} b_{ij}(n) t^n \quad \text{and} \quad A_{ij}(t) = \sum_{n=1}^{\infty} a_{ij}(n) t^n.$$

Then, by (1)

$$\begin{aligned} B_{ij}(t) &= b_{ij}(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{r=1}^{K} a_{ir}(k) b_{rj}(n-k) t^n \\ &= b_{ij}(0) + \sum_{r=1}^{K} \sum_{k=1}^{\infty} a_{ir}(k) t^k \sum_{n=k}^{\infty} b_{rj}(n-k) t^{n-k} \\ &= b_{ij}(0) + \sum_{r=1}^{K} A_{ir}(t) B_{rj}(t), \end{aligned}$$

so that, with the matrix notation

$$B(t) = (B_{ij}(t))_{K \times K} \quad \text{and} \quad A(t) = (A_{ij}(t))_{K \times K},$$

we have

$$B(t) = I + A(t) B(t),$$

or

$$B(t) = (I - A(t))^{-1} = I + \sum_{k=1}^{\infty} A^k(t),$$

which is well defined as long as the largest eigenvalue, $\lambda_1(A(t))$, is less than 1.

Theorem 1. *For a weakly regular lattice,*

$$\mu \geqslant \frac{1}{t_0},$$

where $t_0 = \sup\{t : \lambda_1(A(t)) < 1\}$*, and* $A(t)$ *is the matrix generating function for irreducible bridges on the lattice.*

For practical computations, we usually use a truncated version, $A_N(t)$, of $A(t)$, only considering bridges of length $\leqslant N$. Then, component-wise, $0 \leqslant A_N(t) \leqslant A(t)$, for all $t > 0$, so that $t_0 < t_1$, where $t_1 = \sup\{t : \lambda_1(A_N(t)) < 1\}$. This gives the following useful result, which will be used to get lower bounds for μ on weakly regular lattices.

Corollary 1. *For a weakly regular lattice,*

$$\mu \geqslant \frac{1}{t_1},$$

where $t_1 = \sup\{t : \lambda_1(A_N(t)) < 1\}$*, and* $A_N(t)$ *is the truncated matrix generating function for irreducible bridges on the lattice.*

Remark 2. It is possible to obtain lower bounds for lattices with multiple vertex classes without using the matrix method described above. Consider the generating function

$$A_{ii}(t) = \sum_{k=1}^{\infty} a_{ii}(n)t^n,$$

and let $t_i = \sup\{t : A_{ii}(t) < 1\}$. Then, $\mu \geqslant 1/t_i$, for all i. As above, we can also use truncated versions of the generating functions, but we will get poorer bounds than by using the corollary. As an example, counting irreducible bridges of length at most 4 on the Bow-tie lattice, with two vertex classes, see figure 7, gives

$$A_{11}(t) = 4t^4, \qquad A_{12}(t) = 2t,$$
$$A_{21}(t) = 2t + 4t^2 + 4t^3 + 4t^4, \qquad A_{22}(t) = 4t^4.$$

The simplified method gives a lower bound $\mu \geqslant 1/t$, where $4t^4 = 1$, i.e. $\mu \geqslant \sqrt{2} \approx 1.4142$, whereas corollary 1 gives $\mu \geqslant 2.9662$.

3.3. Lattice representation

When applying the method, the results may depend on which representation of the lattice is used. In order to simplify the computations, we have chosen to use representations where the nodes all have integer coordinates. The same representation was used in the computations leading to upper bounds, but that method does not depend on which representation we choose.

As an example, the $(3^3.4^2)$ lattice, see figure 1, was represented as in figure 4 (left). When applying Kesten's method we need to treat this semi-regular lattice as having two node classes, marked 1 and 2 in the figure. If we are only interested in the number of self-avoiding walks, all vertices are equivalent. The dual of the $(3^3.4^2)$ lattice, figure 4 (right), has three node classes, denoted 1, 2 and 3 in the figure, but in practice only two vertex classes because of vertical symmetry.

Representations for the remaining lattices with degree 5: $(3^2.4.3.4)$, $(3^4.6)$ and Bow-tie (with average degree 5), and their duals, all having average degree 10/3, are given in figures 5–7.

The lattices with degree 3: (3.12^2), $(4.6.12)$, (4.8^2), (6^3), and their duals, all having average degree 6, are shown in figures 8–11. Note that the hexagonal lattice (6^3), see

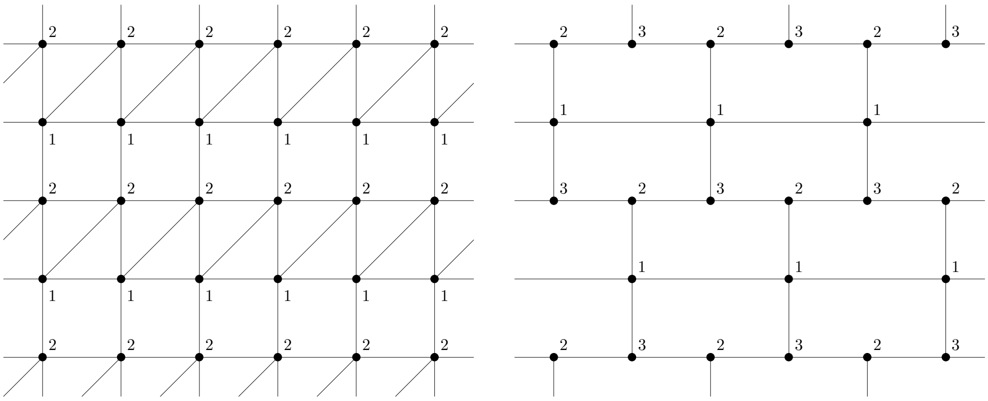

Figure 4. Representation of the $(3^3.4^2)$ lattice and its dual.

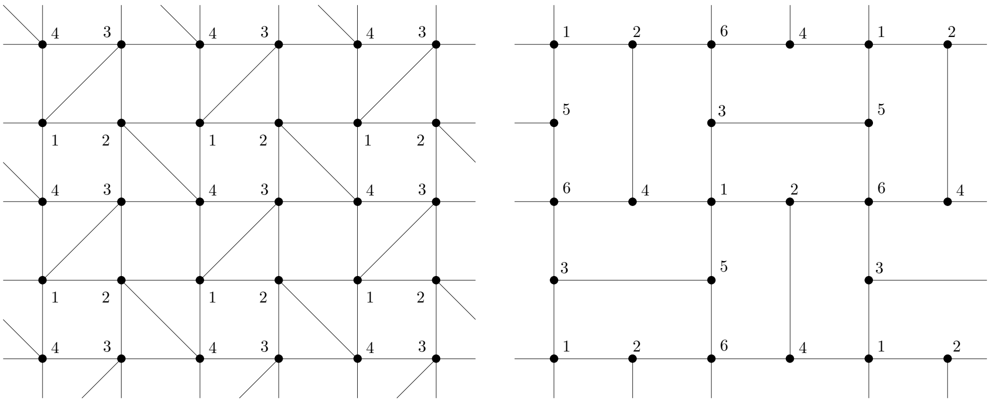

Figure 5. Representation of the $(3^2.4.3.4)$ lattice and its dual.

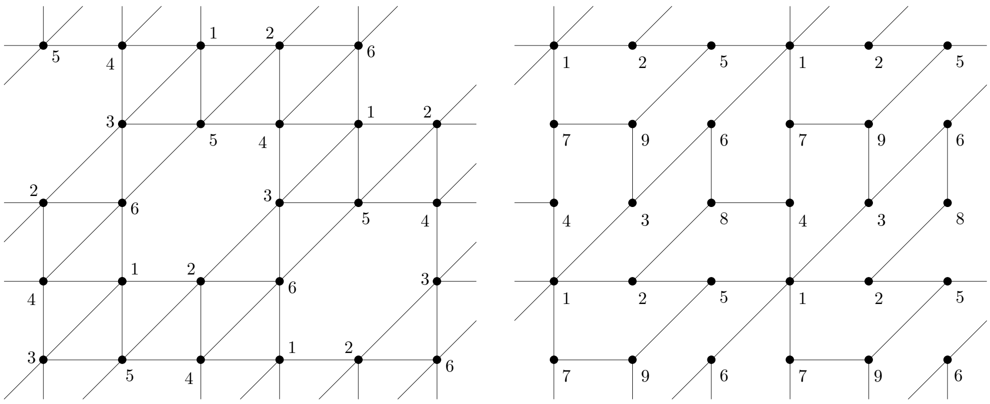

Figure 6. Representation of the $(3^4.6)$ lattice and its dual.

figure 11, although regular, has two vertex classes. Nevertheless, it can be handled with Kesten's original method as all bridges must start (and end) in vertex class 1.

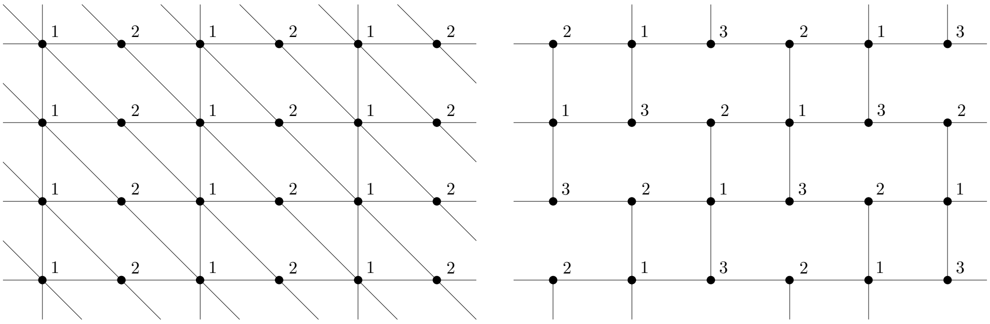

Figure 7. Representation of the Bow-tie lattice and its dual.

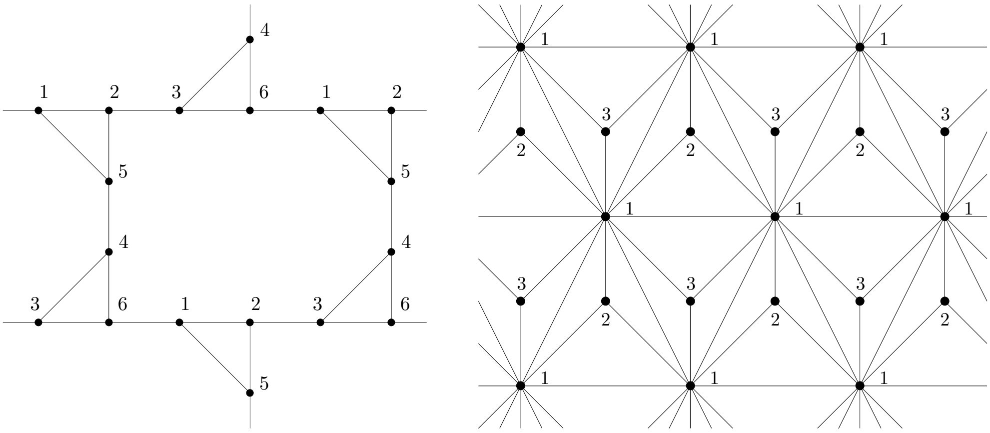

Figure 8. Representation of the (3.12^2) lattice and its dual.

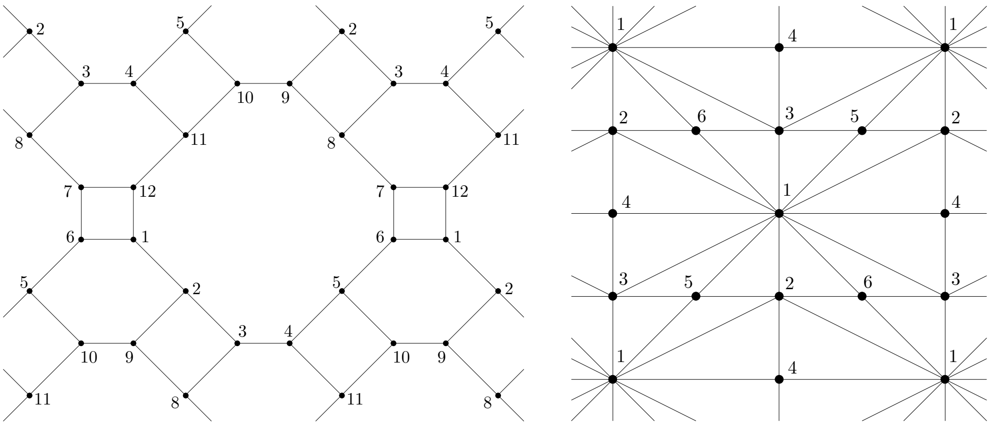

Figure 9. Representation of the (4.6.12) lattice and its dual.

There are five lattices with average degree 4. For the square lattice we use the natural representation. The Kagomé lattice (3.6.3.6) and its dual, also called the Dice lattice, are shown in figure 12. The Ruby lattice (3.4.6.4) and its dual are shown in figure 13.

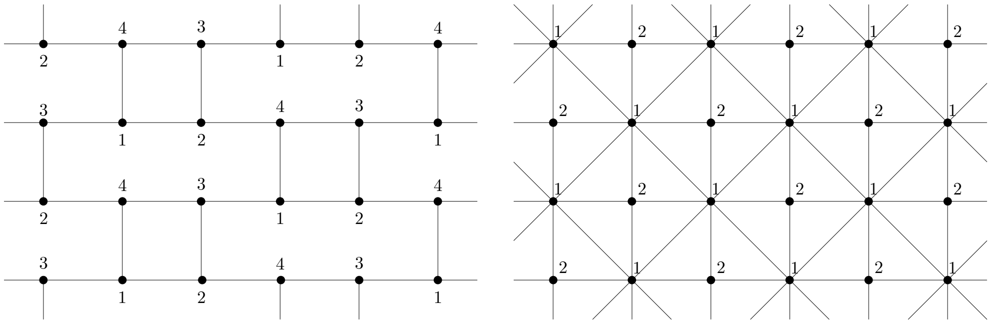

Figure 10. Representation of the (4.8^2) lattice and its dual.

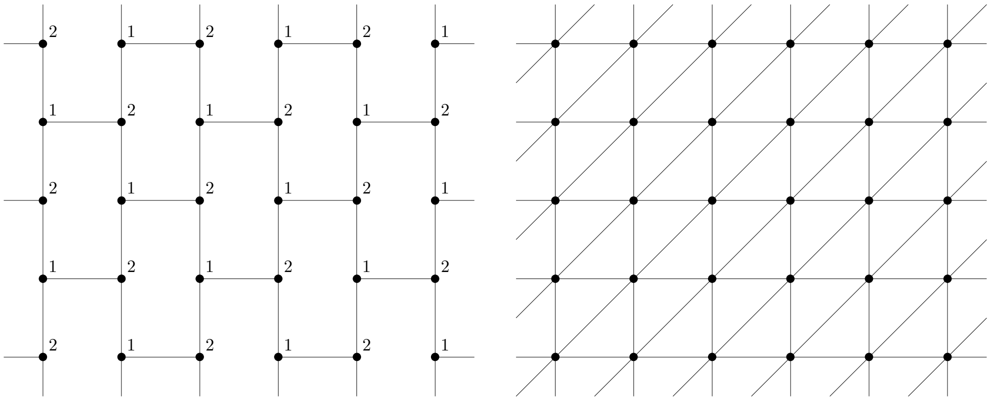

Figure 11. Representation of the hexagonal, (6^3), lattice and its dual, the triangular lattice, (3^6).

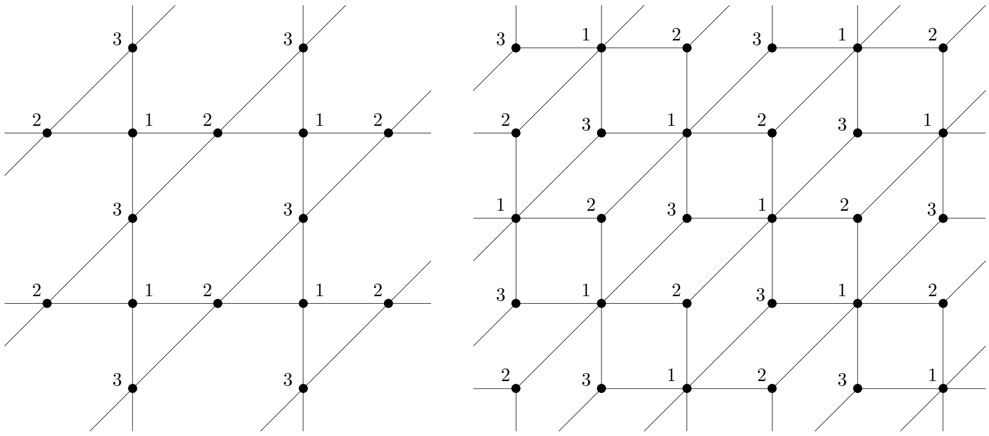

Figure 12. Representation of the (3.6.3.6) lattice and its dual.

Remark 3. We do not claim that the chosen representations are the optimal ones for producing lower bounds. For example, the Kagomé lattice (3.6.3.6) in figure 12 or the (3.12^2) lattice of figure 8 can probably be represented in a more effective way, but we have chosen not to investigate this further as there are better lower bounds available for these lattices, [9].

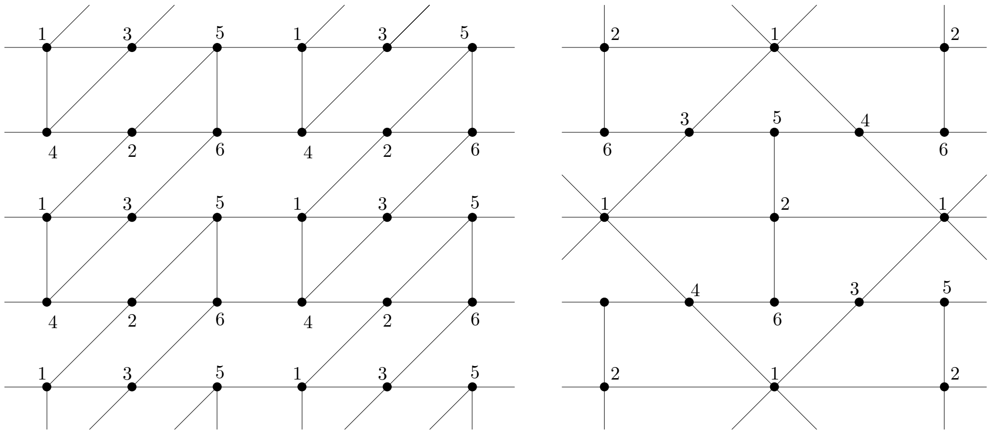

Figure 13. Representation of the (3.4.6.4) lattice and its dual.

Remark 4. When applying corollary 1, the dimension of the matrix $A_N(t)$ may be reduced by removing rows and columns corresponding to vertex classes that cannot be the starting points of bridges, like vertex class 5 in the $D(3^2.4.3.4)$ lattice in figure 5. It is also possible to use vertical symmetry to reduce the dimension. For example, in the (4.8^2) lattice in figure 10, the vertex classes 1 and 4, and the vertex classes 2 and 3, are equivalent, reducing the dimension of the matrix from 4 to 2. An even more significant reduction is obtained for the (4.6.12) lattice, see figure 9, where vertical symmetry reduces the number of vertex classes from 12 to 6.

4. Upper bounds

Improved upper bounds are obtained by the method of Alm [1]. Let

$$F(m) = \sum_{i=1}^{K} f_i(m)$$

be the total number of self-avoiding walks of length m and let $\gamma_i(m)$, $i = 1, \dots, F(m)$, denote these walks. Further, let $g_{ij}(m, n)$ be the number of n-stepped self-avoiding walks that start with $\gamma_i(m)$ and end with (a translation of) $\gamma_j(m)$.

Theorem 2 (Alm 1993). *With*

$$\mathbf{G}(m, n) = (g_{ij}(m, n))_{F(m)\times F(m)},$$

$$\mu \leqslant (\lambda_1(\mathbf{G}(m, n)))^{1/(n-m)},$$

where λ_1 *denotes the largest eigenvalue.*

Remark 5. When using this method, available computer memory limits the choice of m, whereas computing time limits n.

Remark 6. It is possible to reduce the order of $\mathbf{G}(m, n)$ by using more symmetry (reflection and rotation). This has, to some extent, been used in the computations.

5. Results

The methods of the previous sections, theorem 2 for upper bounds and corollary 1 for lower bounds, were used to get bounds for all ALB lattices, improving existing bounds for most of the lattices. The computations extend previous enumerations on all lattices, except the square, triangular and hexagonal. Estimated values were obtained using Domb and Sykes' alpha and Neville tables; see [15], which give a precision of three to four decimal places for these series.

In the following subsections we will group the lattices according to their average degree. A summary of the best available bounds is given in table 1.

The results are discussed in more detail in the following section.

5.1. Degree 3 lattices

There are four ALB lattices with degree 3: (3.12^2), $(4.6.12)$, (4.8^2) and the hexagonal (6^3), all Archimedean; see figures 1 and 8–11.

5.1.1. The (3.12^2) *lattice.* This semi-regular lattice, also known as the *Star* or *extended Kagomé* lattice, has six vertex classes when computing lower bounds; see figure 8.

The matrix $G(18, 48)$, with dimension 23 976, was computed, giving the upper bound $\mu < 1.729\,220$. This does not improve the bound $\mu < 1.719\,254$ obtained in [3] using a relation between $\mu_{(3.12^2)}$ and μ_{HEX} given by Jensen and Guttmann [12],

$$\frac{1}{\mu_{\mathrm{HEX}}} = \frac{1}{\mu_{(3.12^2)}} + \frac{1}{\mu^3_{(3.12^2)}}. \tag{2}$$

This relation was also used by Jensen [9] to obtain the lower bound $\mu > 1.708\,758$ (erroneously given as $\mu > 1.708\,553$ in the paper). Irreducible bridges of length $N \leqslant 53$ only gives $\mu > 1.691\,580$.

The values of $f(n)$ for $n \leqslant 51$ are given in table 2. This extends the enumeration ($n \leqslant 26$) given in [5].

Relation (2) and Nienhuis' supposed value for $\mu_{\mathrm{HEX}} = \sqrt{2+\sqrt{2}}$, determines $\mu_{(3.12^2)} \approx 1.711\,041$, [12].

5.1.2. The $(4.6.12)$ *lattice.* This semi-regular lattice, sometimes referred to as the *Cross* lattice, has six vertex classes when computing lower bounds; see figure 9 and note that by vertical symmetry we need only consider vertex classes 1–6.

The matrix $G(18, 39)$, with dimension 111 702, gives the bound $\mu < 1.809\,064$.

Using irreducible bridges of length $N \leqslant 48$ gives the lower bound $\mu > 1.763\,766$.

Enumeration of self-avoiding walks up to length 47, see table 2, was used to estimate $\mu \approx 1.7871$. We are not aware of any previously published enumeration of self-avoiding walks on this lattice, nor any bounds for or estimate of the connective constant.

5.1.3. The (4.8^2) *lattice.* This semi-regular lattice, also known as the *Bathroom tiling* or *Briarwood* lattice, has two vertex classes when computing lower bounds; see figure 10 and note that by vertical symmetry we need only consider vertex classes 1 and 2.

The matrix $G(19, 42)$, with dimension 125 094, gives the bound $\mu < 1.829\,254$, improving the bound in [1].

Using irreducible bridges of length $N \leqslant 49$ gives the lower bound $\mu > 1.785\,641$. This was recently substantially improved by Jensen [9] to $\mu > 1.804\,596$.

Table 2. Number of self-avoiding walks on the degree 3 lattices.

n	(3.12^2)	$(4.6.12)$	(4.8^2)
1	3	3	3
2	6	6	6
3	10	12	12
4	18	22	22
5	32	42	42
6	56	78	80
7	100	146	152
8	176	264	284
9	312	490	536
10	552	894	988
11	976	1646	1848
12	1724	3012	3412
13	3018	5528	6352
14	5240	10 086	11 724
15	9078	18 476	21 718
16	15 780	33 648	39 952
17	27 502	61 472	73 808
18	47 952	111 702	135 668
19	83 602	203 552	250 188
20	145 700	368 872	459 172
21	253 666	670 538	844 888
22	440 696	1 213 118	1 548 608
23	763 624	2 201 208	2 845 186
24	1 321 176	3 980 380	5 211 548
25	2 286 260	7 214 200	9 563 768
26	3 959 928	13 044 916	17 501 272
27	6 861 692	23 627 064	32 079 524
28	11 886 772	42 714 902	58 660 712
29	20 581 946	77 316 682	107 425 356
30	35 619 908	139 695 536	196 320 596
31	61 607 416	252 664 214	359 232 144
32	106 477 892	456 138 008	656 099 656
33	183 923 972	824 332 804	1 199 676 412
34	317 633 956	1 487 051 098	2 189 995 764
35	548 571 760	2 685 425 808	4 001 911 076
36	947 415 036	4 841 707 570	7 302 060 948
37	1635 944 498	8 738 393 638	13 335 944 432
38	2824 074 824	15 749 389 392	24 322 985 128
39	4873 843 408	28 411 849 334	44 399 312 952
40	8 409 396 972	51 193 846 536	80 948 266 996
41	14 505 967 988	92 317 763 708	147 696 743 656
42	25 015 863 884	166 297 813 974	269 184 560 468
43	43 131 830 640	299 772 356 362	490 946 387 696
44	74 358 090 656	539 832 416 602	894 489 206 772
45	128 179 084 208	972 751 189 854	1 630 785 451 464
46	220 928 082 152	1751 174 705 274	2970 377 146 028
47	380 728 998 492	3154 402 628 922	5413 585 017 968
48	656 014 489 036		
49	1130 187 139 044		
50	1946 827 025 444		
51	3353 058 928 428		

Computation of $f(n)$ up to length 47, see table 2, extends the previous enumeration ($n \leqslant 29$) of [1].

The estimate $\mu \approx 1.809$ agrees with the estimate $\mu \approx 1.808\,830$ given in [12].

5.1.4. The hexagonal lattice. This lattice was treated separately in [3], giving the upper bound $\mu < 1.868\,832$.

The lower bound of that paper, $\mu > 1.833\,009$, was recently improved by Jensen [9] to $\mu > 1.841\,925$.

Enumeration of self-avoiding walks up to length 100 is given by Jensen [11].

The supposed exact value, $\mu = \sqrt{2+\sqrt{2}} \approx 1.847\,759$, of Nienhuis [16] is supported by all extrapolations.

5.2. Lattices with degree 10/3

There are four ALB lattices with average degree 10/3, all duals of lattices with degree 5 and all having two or three vertex classes: D(Bow-tie), $D(3^2.4.3.4)$, $D(3^3.4^2)$ and $D(3^4.6)$, see figures 2–7.

To our knowledge, self-avoiding walks on these lattices have not been studied before.

5.2.1. The dual Bow-tie lattice. This lattice has two vertex classes, one with degree 4 and one with degree 3; see figure 7.

The matrix $G(11, 33)$, with dimension 18 742, gives the bound $\mu < 2.145\,304$.

Using irreducible bridges of length $N \leqslant 38$ gives the lower bound $\mu > 2.076\,706$.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 38$ are given in table 3. Extrapolation of these series gave the estimate $\mu \approx 2.111$.

5.2.2. The dual $(3^2.4.3.4)$ *lattice.* This lattice has two vertex classes, one with degree 4 and one with degree 3; see figure 5.

The matrix $G(10, 32)$, with dimension 31 736, gives the bound $\mu < 2.168\,320$.

Using irreducible bridges of length $N \leqslant 36$ gives the lower bound $\mu > 2.092\,579$.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 36$ are given in table 4. Extrapolation of these series gave the estimate $\mu \approx 2.132$.

5.2.3. The dual $(3^3.4^2)$ *lattice.* This lattice, also known as the *Pentagonal* lattice, has two vertex classes, one with degree 4 and one with degree 3; see figure 4.

The matrix $G(11, 33)$, with dimension 20 743, gives the bound $\mu < 2.186\,720$.

Using irreducible bridges of length $N \leqslant 36$ gives the lower bound $\mu > 2.112\,899$.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 36$ are given in table 4. Extrapolation of these series gave the estimate $\mu \approx 2.152$.

5.2.4. The dual $(3^4.6)$ *lattice.* This lattice has three vertex classes, one with degree 6 and two with degree 3; see figure 6.

The matrix $G(9, 30)$, with dimension 29 784, gives the bound $\mu < 2.235\,067$.

Using irreducible bridges of length $N \leqslant 35$ gives the lower bound $\mu > 2.154\,816$.

The values of $f_1(n)$, $f_2(n)$ and $f_3(n)$ for $n \leqslant 34$ are given in table 5. Extrapolation of these series gave the estimate $\mu \approx 2.155$.

Table 3. Number of self-avoiding walks on the dual Bow-tie lattice.

n	$f_1(n)$	$f_2(n)$
1	4	3
2	8	8
3	20	18
4	44	40
5	96	92
6	220	200
7	476	452
8	1048	974
9	2296	2156
10	4952	4676
11	10 836	10 184
12	23 368	22 034
13	50 688	47 868
14	109 424	103 070
15	235 944	223 300
16	508 280	479 572
17	1094 236	1035 398
18	2349 948	2221 468
19	5052 304	4781 968
20	10 832 340	10 247 458
21	23 246 096	22 018 346
22	49 790 232	47 122 356
23	106 677 536	101 099 276
24	228 257 516	216 139 174
25	488 498 740	463 099 208
26	1044 174 832	989 246 448
27	2232 566 700	2117 080 154
28	4768 148 288	4519 080 698
29	10 186 068 856	9661 885 548
30	21 739 381 308	20 610 366 890
31	46 405 768 288	44 028 642 894
32	98 978 466 556	93 865 037 902
33	211 144 331 144	200 370 472 494
34	450 092 221 988	426 949 915 216
35	959 595 749 204	910 800 515 376
36	2044 514 304 536	1939 831 638 482
37	4356 629 794 320	4135 796 238 488
38	9278 021 270 984	8804 737 338 186

5.3. Lattices with degree 4

There are five ALB lattices with degree 4, three Archimedean: the Kagomé (3.6.3.6), the Ruby (3.4.6.4) and the square (4^4) lattices, and two Laves lattices: the dual Ruby lattice $D(3.4.6.4)$ and the dual Kagomé, or *Dice*, lattice, $D(3.6.3.6)$, see figures 1, 2, 12, 13. Self-avoiding walks on the square lattice have obtained much attention, and the Kagomé lattice has also been studied, but we are not aware of any previous work on the remaining three lattices.

5.3.1. The Kagomé (3.6.3.6) *lattice.* This lattice is semi-regular with two vertex classes when computing lower bounds; see figure 12 and note that vertices denoted 1 and 3 are equivalent.

Table 4. Number of self-avoiding walks on the $D(3^2.4.3.4)$ and $D(3^3.4^2)$ lattices.

	$D(3^2.4.3.4)$		$D(3^3.4^2)$	
n	$f_1(n)$	$f_2(n)$	$f_1(n)$	$f_2(n)$
1	4	3	4	3
2	8	8	10	7
3	20	18	22	18
4	48	42	50	44
5	100	96	114	96
6	232	212	262	218
7	524	478	590	500
8	1124	1064	1302	1114
9	2516	2332	2898	2500
10	5552	5158	6450	5570
11	12 068	11 350	14 254	12 298
12	26 564	24 790	31 474	27 264
13	58 040	54 292	69 402	60 286
14	126 212	118 616	152 730	132 712
15	275 512	258 142	335 818	292 184
16	599 248	562 308	737 254	642 518
17	1300 932	1223 086	1616 318	1410 076
18	2826 440	2654 672	3540 838	3092 262
19	6129 280	5761 760	7750 150	6774 934
20	13 278 992	12 493 394	16 948 422	14 826 488
21	28 764 800	27 057 900	37 038 042	32 424 722
22	62 244 248	58 580 814	80 888 886	70 863 618
23	134 605 624	126 739 248	176 546 146	154 753 446
24	290 981 560	273 998 026	385 107 986	337 755 836
25	628 605 512	592 110 592	839 617 690	736 776 920
26	1357 322 032	1278 871 048	1829 652 318	1606 306 942
27	2929 662 720	2760 749 638	3985 289 798	3500 340 982
28	6320 447 548	5957 414 590	8677 029 278	7624 366 236
29	13 630 470 352	12 850 090 786	18 884 819 642	16 600 123 562
30	29 384 715 412	27 706 609 062	41 086 175 578	36 128 343 180
31	63 324 897 888	59 719 052 078	89 357 421 374	78 601 218 692
32	136 423 406 380	128 675 134 890	194 279 098 870	170 946 755 816
33	293 814 174 776	277 164 772 498	422 269 358 002	371 665 076 262
34	632 599 393 128	596 836 230 624	917 548 489 474	807 815 755 648
35	1361 657 136 640	1284 842 203 420	1993 202 223 970	1755 289 052 740
36	2930 188 540 020	2765 216 402 546	4328 750 731 262	3813 002 741 096

The matrix $G(11, 29)$, with dimension 21 352, gives the bound $\mu < 2.605\,069$. This was improved to $\mu < 2.590\,305$ in [5], by using the fact that the Kagomé lattice is the covering lattice of the hexagonal lattice.

Using irreducible bridges of length $N \leqslant 31$ gives the lower bound $\mu > 2.509\,674$. This was improved to $\mu > 2.548\,497$ in [9].

The values of $f(n)$ for $n \leqslant 31$ are given in table 6. This extends the enumeration in [14] and also corrects an error in their value of $f(28)$. Extrapolation gave the estimate $\mu \approx 2.561$, agreeing with the estimate 2.560 577 by Jensen [9].

5.3.2. The Ruby (3.4.6.4) *lattice.* This lattice is semi-regular with three vertex classes when computing lower bounds; see figure 13 and note that, by vertical symmetry, we need only consider the odd numbered vertices.

Table 5. Number of self-avoiding walks on the $D(3^4.6)$ lattice.

n	$f_1(n)$	$f_2(n)$	$f_3(n)$
1	6	3	3
2	12	9	6
3	24	21	21
4	66	48	51
5	156	117	96
6	336	273	249
7	774	618	621
8	1812	1428	1311
9	4092	3283	2997
10	9078	7420	7107
11	20 556	16 772	15 903
12	46 758	37 949	35 400
13	104 226	85 556	80 508
14	232 314	192 062	182 148
15	523 416	430 654	406 803
16	1171 686	966 247	909 324
17	2606 208	2162 715	2043 768
18	5822 382	4830 079	4575 438
19	13 015 062	10 791 648	10 192 530
20	28 972 326	24 093 622	22 755 822
21	64 467 552	53 698 772	50 843 091
22	143 613 426	119 627 969	113 223 366
23	319 518 462	266 423 930	251 935 404
24	709 905 276	592 793 022	561 093 111
25	1577 405 796	1318 077 102	1248 305 718
26	3504 521 148	2929 772 569	2773 646 481
27	7779 397 464	6509 025 111	6162 825 489
28	17 260 601 976	14 453 142 573	13 690 705 833
29	38 293 410 108	32 080 257 806	30 390 253 506
30	84 923 674 728	71 181 236 128	67 428 989 712
31	188 244 286 188	157 879 264 103	149 585 647 773
32	417 163 852 824	350 046 010 436	331 719 188 994
33	924 267 479 580	775 878 576 160	735 287 624 418
34	2047 120 032 414	1719 234 908 660	1629 405 631 977

The matrix $G(10, 28)$, with dimension 17 113, gives the bound $\mu < 2.610\,835$.

Using irreducible bridges of length $N \leqslant 30$ gives the lower bound $\mu > 2.511\,254$.

The values of $f(n)$ for $n \leqslant 30$ are given in table 6. Extrapolation gave the estimate $\mu \approx 2.564$.

To our knowledge, these are the first results on self-avoiding walks for the Ruby lattice.

5.3.3. The square (4^4) *lattice.* This regular lattice is by far the most studied of the ALB lattices in connection with self-avoiding walks.

The best upper bound, $\mu < 2.679\,193$, was given by Pönitz and Tittman [18].

The best lower bound, $\mu > 2.625\,622$, was obtained by Jensen [9], by computing irreducible bridges of length $N \leqslant 72$.

Enumeration up to length 71 was produced by Jensen [8], who in [7] gave the estimate $\mu \approx 2.638\,159$ based on self-avoiding polygons.

Table 6. Number of self-avoiding walks on the Kagomé (3.6.3.6), and Ruby (3.4.6.4), lattices.

n	Kagomé	Ruby
1	4	4
2	12	12
3	32	34
4	88	94
5	240	252
6	652	680
7	1744	1826
8	4616	4858
9	12 208	12 928
10	32 328	34 226
11	85 408	90 298
12	224 640	237 710
13	589 024	624 318
14	1542 944	1637 370
15	4039 256	4289 652
16	10 560 552	11 226 044
17	27 567 488	29 347 138
18	71 878 068	76 636 640
19	187 262 944	199 927 120
20	487 526 944	521 101 204
21	1268 269 160	1357 191 780
22	3296 832 292	3532 445 834
23	8564 411 120	9188 678 794
24	22 235 825 104	23 888 535 986
25	57 701 041 072	62 072 114 752
26	149 657 337 872	161 207 840 658
27	387 978 891 176	418 478 353 298
28	1005 378 745 536	1085 857 527 206
29	2604 222 063 144	2816 439 313 010
30	6743 181 213 712	7302 441 586 124
31	17 454 178 002 264	

5.3.4. The dual Ruby lattice $D(3.4.6.4)$. This lattice has three vertex classes, one with degree 6, one with degree 4 and one with degree 3. When computing lower bounds, we need to consider the four vertex classes 1, 2, 3 and 5 of figure 13.

The matrix $G(7, 24)$, with dimension 18 876, gives the bound $\mu < 2.828\,174$.

Using irreducible bridges of length $N \leqslant 28$ gives the lower bound $\mu > 2.693\,424$.

The values of $f_1(n)$, $f_2(n)$ and $f_3(n)$ for $n \leqslant 28$ are given in table 7. Extrapolation of these series gave the estimate $\mu \approx 2.763$.

5.3.5. The Dice lattice $D(3.6.3.6)$. This lattice, the dual of the Kagomé lattice, has two vertex classes, one with degree 6 and one with degree 3. When computing lower bounds, we need only consider the two vertex classes denoted 1 and 2 of figure 12 due to vertical symmetry.

The matrix $G(9, 25)$, with dimension 24 224, gives the bound $\mu < 2.817\,739$.

Using irreducible bridges of length $N \leqslant 30$ gives the lower bound $\mu > 2.704\,239$.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 29$ are given in table 8. Extrapolation of these series gave the estimate $\mu \approx 2.761$.

Table 7. Number of self-avoiding walks on the dual Ruby, $D(3.4.6.4)$, lattice.

n	$f_1(n)$	$f_2(n)$	$f_3(n)$
1	6	4	3
2	18	14	9
3	54	42	36
4	150	130	102
5	474	370	318
6	1302	1130	882
7	3954	3146	2742
8	10 734	9490	7512
9	32 370	26 006	22 824
10	87 426	77 926	61 962
11	261 894	211 726	186 642
12	704 454	631 614	504 030
13	2101 902	1706 510	1508 814
14	5638 086	5074 578	4058 706
15	16 768 290	13 656 330	12 100 350
16	44 872 206	40 505 102	32 454 966
17	133 104 294	108 671 358	96 454 890
18	355 506 570	321 657 702	258 097 140
19	1052 388 198	860 905 450	765 164 076
20	2806 489 962	2544 046 834	2043 592 116
21	8294 540 826	6796 085 402	6046 857 690
22	22 091 343 810	20 056 146 286	16 125 110 496
23	65 202 978 942	53 493 772 878	47 639 169 846
24	173 468 654 478	157 688 602 514	126 875 692 236
25	511 412 880 042	420 037 285 362	374 349 488 022
26	1359 302 432 034	1236 987 348 006	995 901 463 296
27	4003 554 217 410	3291 327 878 982	2935 214 709 768
28	10 632 501 183 834	9684 733 628 410	7801 379 718 852

5.4. Lattices with degree 5

There are four ALB lattices with degree 5: the three semi-regular $(3^4.6)$, $(3^3.4^2)$ and $(3^2.4.3.4)$, see figures 1 and 4–6, and the weakly regular Bow-tie lattice, with average degree 5, see figures 3 and 7. To our knowledge, none of these have been studied in connection with self-avoiding walks before.

5.4.1. The $(3^4.6)$ *lattice.* This semi-regular lattice has six vertex classes when computing bridges; see figure 6.

The matrix $G(7, 22)$, with dimension 10 372, gives the upper bound $\mu < 3.369\,117$.

Using irreducible bridges of length $N \leqslant 24$, we get the lower bound $\mu > 3.206\,403$.

The values of $f(n)$ for $n \leqslant 23$ are given in table 9. Extrapolation gave the estimate $\mu \approx 3.293$.

5.4.2. The $(3^3.4^2)$ *lattice.* This semi-regular lattice has two vertex classes when computing bridges; see figure 4.

The matrix $G(9, 21)$, with dimension 70 883, gives the upper bound $\mu < 3.425\,364$.

Using irreducible bridges of length $N \leqslant 24$, we get the lower bound $\mu > 3.266\,402$.

The values of $f(n)$ for $n \leqslant 23$ are given in table 9. Extrapolation gave the estimate $\mu \approx 3.350$.

Table 8. Number of self-avoiding walks on the Dice, $D(3.6.3.6)$, lattice.

n	$f_1(n)$	$f_2(n)$
1	6	3
2	12	15
3	60	30
4	108	144
5	528	264
6	912	1266
7	4428	2214
8	7512	10 632
9	36 336	18 168
10	61 056	87 276
11	294 588	147 294
12	491 280	706 992
13	2365 104	1182 552
14	3923 232	5672 628
15	18 862 128	9431 064
16	31 159 248	45 213 792
17	149 642 496	74 821 248
18	246 387 456	358 519 356
19	1182 286 308	591 143 154
20	1941 449 952	2831 337 912
21	9309 674 928	4654 837 464
22	15 253 711 488	22 286 434 278
23	73 104 036 204	36 552 018 102
24	119 556 045 792	174 946 751 040
25	572 709 412 368	286 354 706 184
26	935 130 657 696	1370 172 679 248
27	4477 780 172 100	2238 890 086 050
28	7301 340 370 800	10 710 133 253 376
29	34 949 818 263 840	17 474 909 131 920

5.4.3. The $(3^2.4.3.4)$ *lattice.* This semi-regular lattice has two vertex classes when computing bridges; see figure 5 and note the vertical symmetry.

The matrix $G(8, 22)$, with dimension 21 326, gives the upper bound $\mu < 3.451\,433$.

Using irreducible bridges of length $N \leqslant 24$, we get the lower bound $\mu > 3.285\,284$.

The values of $f(n)$ for $n \leqslant 23$ are given in table 9. Extrapolation gave the estimate $\mu \approx 3.374$.

5.4.4. The Bow-tie lattice. This weakly regular lattice has two vertex classes, one with degree 6 and one with degree 4; see figure 7 and note the vertical symmetry.

The matrix $G(8, 22)$, with dimension 25 571, gives the upper bound $\mu < 3.525\,448$.

Using irreducible bridges of length $N \leqslant 25$, we get the lower bound $\mu > 3.357\,574$.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 23$ are given in table 10. Extrapolation gave the estimate $\mu \approx 3.4455$.

5.5. Lattices with degree 6

There are four ALB lattices with degree 6, all duals of the degree 3 lattices: the regular triangular (3^6) lattice and the three Laves lattices $D(4.8^2)$, $D(4.6.12)$ and $D(3.12^2)$. Of these,

Table 9. Number of self-avoiding walks on the $(3^4.6)$, $(3^3.4^2)$ and $(3^2.4.3.4)$ lattices.

n	$(3^4.6)$	$(3^3.4^2)$	$(3^2.4.3.4)$
1	5	5	5
2	20	20	20
3	72	74	74
4	252	266	270
5	874	948	970
6	3016	3344	3440
7	10 372	11 724	12 148
8	35 538	40 850	42 652
9	121 284	141 766	149 100
10	412 242	490 316	519 520
11	1395 976	1691 252	1805 228
12	4713 356	5820 270	6257 724
13	15 882 524	19 991 578	21 649 360
14	53 452 630	68 550 952	74 771 232
15	179 732 292	234 711 768	257 853 108
16	603 784 384	802 581 256	888 050 112
17	2026 136 020	2741 197 536	3054 903 228
18	6791 270 462	9352 835 040	10 497 994 420
19	22 738 287 950	31 881 907 526	36 042 084 224
20	76 060 696 412	108 588 062 224	123 636 733 660
21	254 235 160 722	369 564 222 160	423 792 385 416
22	849 257 603 032	1256 888 408 900	1451 630 772 024
23	2835 303 656 310	4271 966 080 654	4969 151 186 440

Table 10. Number of self-avoiding walks on the Bow-tie lattice.

n	$f_1(n)$	$f_2(n)$
1	6	4
2	22	20
3	86	72
4	318	272
5	1170	1008
6	4230	3676
7	15 226	13 292
8	54 550	47 732
9	194 738	170 684
10	692 890	608 228
11	2458 174	2161 060
12	8700 818	7658 012
13	30 736 794	27 079 364
14	108 402 594	95 579 160
15	381 754 478	336 830 848
16	1342 664 262	1185 394 144
17	4716 828 182	4166 626 488
18	16 553 404 838	14 629 643 560
19	58 039 661 590	51 316 934 576
20	203 330 098 250	179 848 874 136
21	711 788 064 986	629 812 096 608
22	2490 007 793 146	2203 948 469 260
23	8705 161 472 354	7707 365 570 308

Table 11. Number of self-avoiding walks on the $D(4.8^2)$ lattice.

n	$f_1(n)$	$f_2(n)$
1	8	4
2	40	28
3	200	140
4	960	692
5	4528	3316
6	21 192	15 620
7	98 472	73 028
8	455 424	338 972
9	2097 064	1565 908
10	9622 896	7203 772
11	44 037 032	33 032 636
12	201 060 376	151 072 012
13	916 164 480	689 368 412
14	4167 514 720	3139 701 844
15	18 929 322 048	14 276 075 436
16	85 866 898 520	64 819 327 908
17	389 057 491 544	293 934 346 628
18	1760 975 135 408	1331 399 162 948
19	7963 242 558 008	6024 629 806 972

to our knowledge only the triangular has been studied before in connection with self-avoiding walks.

5.5.1. The triangular lattice (3^6). This regular lattice has only one vertex class; see figure 11.

The matrix $G(8, 20)$, with dimension 18 678, gives the upper bound $\mu < 4.251\,419$, which improves the bound in [1].

Using irreducible bridges, Jensen [9] obtained the lower bound $\mu > 4.118\,935$.

In [10], Jensen enumerates self-avoiding walks up to length 40, and uses extrapolation to estimate $\mu \approx 4.150\,797$.

5.5.2. The dual (4.8^2) *lattice.* This weakly regular lattice, also known as the *Octagonal* lattice, has two vertex classes, one with degree 8 and one with degree 4; see figure 10.

The matrix $G(7, 18)$, with dimension 25 748, gives the upper bound $\mu < 4.565\,362$.

Using irreducible bridges of length $N \leqslant 20$, we get the lower bound $\mu > 4.304\,718$. Note that the lower bound exceeds the upper bound for the triangular lattice.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 19$ are given in table 11. Extrapolation gave the estimate $\mu \approx 4.442$.

5.5.3. The dual (4.6.12) *lattice.* This weakly regular lattice has three vertex classes, one with degree 12, one with degree 6 and one with degree 4. When computing bridges we need to consider four vertex classes; see figure 9 and note the vertical symmetry.

The matrix $G(5, 16)$, with dimension 29 916, gives the upper bound $\mu < 4.787\,227$.

Using irreducible bridges of length $N \leqslant 18$, we get the lower bound $\mu > 4.463\,058$.

The values of $f_1(n)$, $f_2(n)$ and $f_3(n)$ for $n \leqslant 18$ are given in table 12. Extrapolation gave the estimate $\mu \approx 4.624$.

Table 12. Number of self-avoiding walks on the $D(4.6.12)$ lattice.

n	$f_1(n)$	$f_2(n)$	$f_3(n)$
1	12	6	4
2	48	42	32
3	288	198	160
4	1344	1068	852
5	6828	5196	4232
6	32 892	25 902	21 020
7	159 612	125 874	102 652
8	766 356	609 780	497 720
9	3671 076	2933 562	2397 844
10	17 521 560	14 058 132	11 501 012
11	83 440 932	67 139 772	54 967 576
12	396 541 656	319 822 572	261 998 092
13	1881 162 084	1520 161 374	1245 969 948
14	8909 612 856	7211 880 744	5913 866 044
15	42 136 382 208	34 157 352 042	28 021 308 344
16	199 020 641 232	161 541 458 514	132 570 243 968
17	938 971 412 124	763 007 236 542	626 365 075 348
18	4425 660 916 764	3599 867 690 610	2956 008 677 160

Table 13. Number of self-avoiding walks on the $D(3.12^2)$ lattice.

n	$f_1(n)$	$f_2(n)$
1	12	3
2	78	33
3	498	222
4	3030	1410
5	18 102	8 598
6	107 010	51 414
7	627 978	304 032
8	3664 842	1784 232
9	21 292 854	10 411 440
10	123 273 066	60 482 682
11	711 614 178	350 116 536
12	4097 986 746	2020 881 804
13	23 550 744 894	11 636 504 136
14	135 105 470 730	66 867 702 000
15	773 884 996 398	383 573 275 764
16	4426 872 850 098	2196 943 368 528
17	25 293 115 756 146	12 566 359 027 902

5.5.4. The dual (3.12^2) *lattice.* This weakly regular lattice, also known as the *Asanoha* lattice, has two vertex classes, one with degree 12 and one with degree 3; see figure 8 and note the vertical symmetry.

The matrix $G(5, 15)$, with dimension 9493, gives the upper bound $\mu < 5.796\,210$. This was improved in [5] to $\mu < 5.734\,24$, using a relation with the triangular lattice.

Using irreducible bridges of length $N \leqslant 17$, we get the lower bound $\mu > 5.377\,158$. Note that the lower bound exceeds the upper bound for the $D(4.6.12)$ lattice.

The values of $f_1(n)$ and $f_2(n)$ for $n \leqslant 17$ are given in table 13. Extrapolation gave the estimate $\mu \approx 5.595$.

6. Discussion

In table 1 we summarize the best upper and lower bounds for, and estimates of, the connective constants for the ALB lattices. In tables 2–13 we give enumerations for all ALB lattices except the three regular: square, triangular and hexagonal lattices.

6.1. Partial ordering

Table 1 also gives a partial ordering of the ALB lattices with respect to connective constants, with horizontal lines indicating a strict ordering relation; graphs above a line have a strictly higher connective constant than graphs below the line. This notation gives the partial ordering available at the moment with one exception:

$$\mu_{D(3^4.6)} > \mu_{D(\text{Bow-tie})}.$$

To get a complete ordering of the ALB lattices with respect to connective constants, there are 31 remaining relations, out of 210, to decide. Some of these could probably be resolved with current methods, just using more computing time or memory, but some certainly seem to require improved methods.

Remark 7. In [17], Parviainen and Wierman give a complete subgraph partial ordering of the Archimedean and Laves lattices. If G_1 is a subgraph of G_2, then $\mu_{G_1} \leqslant \mu_{G_2}$. Unfortunately, the subgraph partial order does not add any new relations between the connective constants of the ALB lattices.

6.2. Average degree

Let $q(G)$ denote the (average) degree of the lattice G. From table 1 we note the following.

Observation 1. *If G_1 and G_2 are ALB lattices, then $q(G_1) < q(G_2) \Rightarrow \mu_{G_1} < \mu_{G_2}$.*

As the degree q only takes five different values for the 21 ALB lattices, although the known variation in μ is much larger, it is tempting to try to find a more sensitive measure of connectivity. In [2], an alternative measure of average degree, $\tilde{q}$, is introduced, defined as the limit

$$\tilde{q} = \lim_{n\to\infty} g_i(n)^{1/n},$$

where $g_i(n)$ is the number of *walks* of length n, starting in vertex class i, on the lattice. The limit is independent of which vertex class the walks start in for all connected graphs. For regular and semi-regular lattices, $\tilde{q} = q$, and for weakly regular lattices, $\tilde{q}$ is easily calculated as an eigenvalue. For all lattices, $\tilde{q} \geqslant q$.

The values of $\tilde{q}$ for the ALB lattices are given in table 1. Note that $\tilde{q}$ takes 12 different values for the 21 ALB lattices compared to only 5 values for the ordinary average degree, q. The estimated values of the connective constants support the following conjecture.

Conjecture 1. *If G_1 and G_2 are ALB lattices, then $\tilde{q}(G_1) < \tilde{q}(G_2) \Rightarrow \mu_{G_1} < \mu_{G_2}$.*

Remark 8. From table 1 we see that to prove the conjecture, it remains to show the following seven relations:

$$\begin{aligned}\mu_{\text{Bow-tie}} &> \mu_{(3^2.4.3.4)},\\ &> \mu_{(3^3.4^2)},\\ &> \mu_{(3^4.6)},\\ \mu_{D(3^4.6)} &> \mu_{D(3^3.4^2)},\\ &> \mu_{D(3^2.4.3.4)},\\ \mu_{D(3^3.4^2)} &> \mu_{D(3^2.4.3.4)},\\ &> \mu_{D(\text{Bow-tie})}.\end{aligned}$$

Judging from the estimated values, the penultimate inequality is probably hardest to prove, and it is unlikely that it can be proved with currently available methods, at least with today's computers.

6.3. Duality

For critical probabilities, p_c, in bond percolation, the following relation holds:

$$p_c(G) + p_c(D(G)) = 1.$$

This in turn implies that

$$p_c(G_1) < p_c(G_2) \quad \Rightarrow \quad p_c(D(G_1)) > p_c(D(G_2)).$$

The corresponding implication for connective constants holds for all pairs of ALB lattices with one possible exception: the Kagomé (3.6.3.6) and the Ruby (3.4.6.4) lattices, where the estimated values indicate that $\mu_{(3.6.3.6)} < \mu_{(3.4.6.4)}$ and that the same order holds for the duals, $\mu_{D(3.6.3.6)} < \mu_{D(3.4.6.4)}$, although the difference in estimated values is very small (0.002). It would be interesting to have more reliable estimates for these connective constants.

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